

The Witten Index Revisited

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We attempt to deal with the orbifold singularities in the moduli space of flat connections for supersymmetric gauge theories on the torus. The fields are restricted to the fundamental domain, containing no gauge copies, but requiring a boundary condition in field space.

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1. Introduction

We revisit supersymmetric Yang-Mills theories on the torus to study the vacuum state in connection with the Witten index [1]. The torus geometry is crucial to preserve the supersymmetry. The index counts the number of quantum states (fermionic states with a negative sign). Due to supersymmetry, states at non-zero energy occur in fermionic and bosonic pairs, and do not contribute to the Witten index. The counting can therefore be reduced to the vacuum sector.

Witten argued that the wave function is constant on the moduli space of flat connections and the degeneracy due to the constant abelian gluino components costs no energy [1]. Requiring gauge invariance (under Weyl reflections) one has $r + 1$ ($= 2$ for $SU(2)$) invariant vacuum states which are all bosonic, $|0\rangle, U|0\rangle, \dots, U^r|0\rangle$, where $U = \sum_{a=1}^r \bar{\lambda}_\alpha^a \bar{\lambda}_\beta^a \varepsilon^{\alpha\beta}$. Untill 1998 there was a puzzle: It didn't agree with the infinite volume gluino condensate calculations for $SO(N \geq 7)$ and the exceptional groups [2]. (Additional references are contained in [3].) Witten solved that by finding an extra disconnected component in the moduli space for $SO(7)$, using a D-brane orientifold construction [4]. This means that there are isolated points for $SO(7)$ and $SO(8)$ and for $SO(N \geq 9)$ there is an extra component which is like the trivial component for $SO(N-7)$. One therefore finds $1 + r(SO(N)) + 1 + r(SO(N-7)) = h(SO(N))$, or $1 + [N/2] + 1 + [(N-7)/2] = N-2$ (h is the dual Coxeter number). This is precisely the infinite volume result! The field theory construction works also for

Group G	h	Vacuum-type					
		1	2	3	4	5	6
$SU(N)$	N	N					
$Sp(N)$	$N + 1$	$N + 1$					
$SO(2N + 1)$	$2N - 1$	$N + 1$	$N - 2$				
$SO(2N)$	$2N - 2$	$N + 1$	$N - 3$				
G_2	4	3	1				
F_4	9	5	2	(1+1)			
E_6	12	7	3	(1+1)			
E_7	18	8	4	(2+2)	(1+1)		
E_8	30	9	5	(3+3)	(2+2)	(1+1+1+1)	(1+1)

Table 1: Contributions to the index for small volumes (A. Keurentjes, JHEP **05**, 001(1999))

exceptional groups [5, 6], and is based on classifying triples, $\Omega_i = P \exp(\int_0^L A_i dx_i)$, which satisfy $\Omega_i \Omega_j = \Omega_j \Omega_i$. Also here one finds that the index in small volumes equals the large volume results.

But a technical problem occurs when using periodic boundary conditions. This is the breakdown of the adiabatic approximation in the reduction of the degrees of freedom to those of the classical vacuum. For $SU(2)$ the classical vacuum is defined up to a gauge transformation by the set of zero-momentum abelian gauge fields, $A_i = \frac{ic_i^3 \tau_3}{2L}$, where c_i^3 is constant and $\vec{\tau}$ are the Pauli matrices. Its gauge invariant parametrization is in terms of the Wilson loops that wind around the three compact directions of the torus, which are *compact* variables. This describes the vacuum valley as an orbifold, T^3/Z_2 for $SU(2)$. Here $g_{[\vec{n}]} = \exp(-2\pi i \vec{n} \cdot \vec{x} \tau_3 / L)$ (which maps c_i^3 to $c_i^3 + 4\pi n_i$) and $g = i\tau_2$ (which maps c_i^3 to $-c_i^3$) give Gribov copies. The orbifold singularities arise where the flat connection is invariant under (part of) the Weyl group (the remnant gauge transformations that

leave the set of zero-momentum abelian gauge fields invariant). For $SU(2)$ there are eight orbifold singularities (related to $A = 0$ by *anti-periodic* gauge transformations), $c_i^3 = -c_i^3 + 4\pi n_i$, giving $c_i^3 = 2\pi n_i$ (with $n_i = 0$ or 1). It was studied in [3] and we will review some of it here.

2. The Hamiltonian

We choose the dependence on the bare coupling constant such that

$$\mathcal{L} = -\frac{1}{4g_0^2}(F_{\mu\nu}^a)^2 + \frac{i}{2g_0^2}\bar{\lambda}^a\gamma_\mu(D^\mu\lambda)^a. \quad (2.1)$$

The reduction to the zero-momentum degrees of freedom, as in the bosonic case, will replace the bare coupling constant by a running and asymptotically free coupling constant $g(L)$. The zero-momentum gauge fields are parametrized as $A_i = ic_i^a\tau_a/(2L)$. The vacuum valley is parametrized by the abelian degrees of freedom. These are defined by r_i , with $r_i r_j = \sum_a c_i^a c_j^a$, for each i and j . As said, we may also parametrize the vacuum valley by $r_i = C_i \equiv c_i^3$.

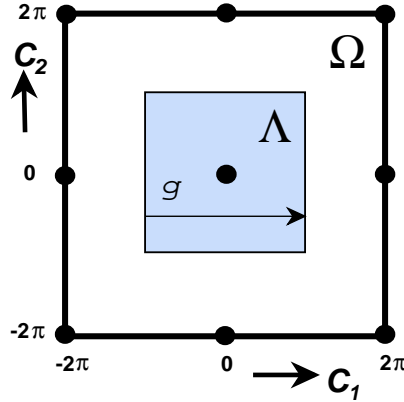


Figure 1: A two dimensional slice of the vacuum valley along the (C_1, C_2) plane. The grey square is the fundamental domain Λ . The dots are gauge copies of the origin (which turn out to lie on the Gribov horizon Ω , indicated by the fat square).

The cell $\vec{C} \in [-\pi, \pi]^3$ can be used as a fundamental domain Λ . Any point on the vacuum valley can be reached by applying suitable gauge transformations. This reduces the eight orbifold singularities to one, but at a cost: Opposite sides on it's boundary are identified by homotopically non-trivial gauge transformations (indicated by g in Fig. 1). The representations of their homotopy define the electric flux quantum numbers as introduced by 't Hooft [7]. We will here only consider the sector with zero electric flux, i.e. the trivial representation, where wave functions at opposite sides are equal. Since the orbifold singularity implies massless excitations for the transverse fluctuations, we seemingly have no choice: We have to keep the non-abelian zero-momentum modes, and thus the adiabatic approximation breaks down.

The energy gap in the fluctuations transverse to the vacuum valley is easily read off. Close to the origin it is given by $2|\vec{C}|/L$. Integrating out transverse degrees of freedom is only reliable if the energy of the low-lying states is smaller than this gap. This energy behaves as $g^{2/3}(L)/L$. Consider now a sphere of radius $g^{1/3}(L)$ around each orbifold, beyond which the adiabatic approximation is

accurate. There, the wave function can be reduced to the vacuum valley and, assuming indeed it has zero energy, it will be constant. As long as $g(L)$ is small, we may assume the wave function in the neighborhood of the orbifold singularity to respect spherical symmetry. In the bosonic case, once the wave function spreads out over the vacuum valley, any spherical symmetry is quickly lost. In the supersymmetric case, however, the reduced wave function on the vacuum valley will become constant up to exponential corrections at distances much greater than $g^{2/3}(L)$ from the orbifold singularities, which are separated over a distance 2π . Instead of insisting that the groundstate wave function is normalizable, we should rather insist on its projection to the vacuum valley to become constant. Spherical symmetry near the orbifold singularities will dramatically simplify the analysis.

We will now set up the zero-momentum supersymmetric Hamiltonian, and its reduction to the gauge invariant, spherically symmetric sector. This is not new [8], but we will be able to push it to the point where we can explicitly construct a complete basis of states that respect these symmetries. We start from the supercharge operators,

$$Q_\alpha = \sigma_{\alpha\dot{\beta}}^j \bar{\lambda}_a^{\dot{\beta}} \left(-i \frac{\partial}{\partial V_j^a} - i B_j^a \right), \quad \bar{Q}_{\dot{\alpha}} = \lambda_a^\beta \sigma_{\beta\dot{\alpha}}^j \left(-i \frac{\partial}{\partial V_j^a} + i B_j^a \right), \quad (2.2)$$

with $V_i^a \equiv c_i^a / (g(L)L)$ and $\sigma^j = \tau^j$ (and σ^0 the unit) as 2×2 matrices. Restricting to the zero-momentum modes, both the Weyl spinors λ_a^β and $\bar{\lambda}_a^{\dot{\beta}}$ are constant. Lowering indices is done with $\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = -i\tau_2$, δ_{ab} and $\eta_{\mu\nu} = \text{diag}(1, 1, 1, -1)$ (or δ_{ij}) for respectively the spinor, group and space-time (or space) indices, and raising of indices is done with the inverse of these matrices. Repeated indices are assumed to be summed over, but to keep notations transparent we will not always balance the positions of the gauge and space indices. Finally, for zero-momentum gauge fields one has $B_i^a = -\frac{1}{2}g\varepsilon_{ijk}\varepsilon_{abc}V_j^bV_k^c$.

In the Hamiltonian formulation the anti-commutation relations $\{\lambda^{a\alpha}, \lambda^{b\beta}\} = 0$, $\{\bar{\lambda}^{a\dot{\alpha}}, \bar{\lambda}^{b\dot{\beta}}\} = 0$, $\{\lambda^{a\alpha}, \bar{\lambda}^{b\dot{\beta}}\} = \bar{\sigma}_0^{\dot{\beta}\alpha} \delta^{ab}$, with $\bar{\sigma}_0$ the unit 2×2 matrix (one has $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\beta} \varepsilon^{\alpha\beta} (\sigma^\mu)_{\beta\dot{\beta}}$), give $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma_0)_{\alpha\dot{\alpha}} \mathcal{H} - 2(\sigma^i)_{\alpha\dot{\alpha}} V_i^a \mathcal{G}_a$, where

$$\mathcal{G}_a = ig\varepsilon_{abc} \left(V_j^c \frac{\partial}{\partial V_j^b} - \bar{\lambda}^b \bar{\sigma}_0 \lambda^c \right) \quad (2.3)$$

is the generator of infinitesimal gauge transformations, and \mathcal{H} is the Hamiltonian density

$$\mathcal{H} = -\frac{1}{2} \frac{\partial^2}{\partial V_i^a \partial V_i^a} + \frac{1}{2} B_i^a B_i^a - ig\varepsilon_{abc} \bar{\lambda}^a \bar{\sigma}^j \lambda^b V_j^c. \quad (2.4)$$

Splitting the Hamiltonian, $\int d^3x \mathcal{H} \equiv g^{2/3}(L)H/L$, in its bosonic and fermionic pieces, $H = H_B + H_f$, we find with $c_i^a = g^{2/3}(L)\hat{c}_i^a$ and $\hat{B}_a^i = -\frac{1}{2}\varepsilon^{ijk}\varepsilon_{abd}\hat{c}_j^b\hat{c}_k^d$ that $H_B = -\frac{1}{2}(\partial/\partial\hat{c}_i^a)^2 + \frac{1}{2}(\hat{B}_i^a)^2$ and $H_f = -i\varepsilon_{abd}\bar{\lambda}^a\bar{\sigma}^i\lambda^b\hat{c}_i^d$.

The orbifold singularities, other than at $\hat{c} = 0$, lie at a distance $b = 2\pi g^{-2/3}(L)$ in these new variables \hat{c} (measured along the vacuum valley where \hat{B} vanishes). We want to solve for the groundstate wave function such that for $|\hat{c}| \gg 1$ it becomes a constant, after projecting to the vacuum valley. As this boundary condition is compatible with spherical symmetry, i.e. it goes to the same constant for all directions on the vacuum valley, we will restrict ourselves to wave functions $\Psi(\hat{c})$ that are spherically symmetric and gauge invariant. We stress again that this is an accidental spherical symmetry, that holds in sufficiently small volumes.

3. Vacuum valley and boundary condition

States with odd fermion number F do not respect the symmetry and particle-hole duality relates $F = 0$ to $F = 6$ and $F = 2$ to $F = 4$ [3]. Without fermions ($F = 0$) this will reduce to the bosonic Hamiltonian which is known to have a non-zero vacuum energy [9]. We therefore write down the most general $F = 2$ states, that is the two-spinor states which are symmetric or antisymmetric in the gauge index (and thus respectively antisymmetric and symmetric in the spinor index), $|\mathcal{V}\rangle \equiv \mathcal{V}_j^a \mathcal{J}_a^j \equiv -2i \mathcal{V}_j^c \varepsilon_{abc} \bar{\lambda}_\alpha^a (\bar{\sigma}^{j0})_\beta^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}b} |0\rangle$ and $|\mathcal{S}\rangle \equiv \mathcal{S}_{ab} \mathcal{J}^{ab} \equiv -\mathcal{S}_{ab} \bar{\lambda}_\alpha^a \bar{\lambda}_\beta^b \varepsilon^{\dot{\alpha}\dot{\beta}} |0\rangle$ (where $\bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$, such that $\bar{\sigma}^{j0} = \frac{1}{2} \tau_j$ as a 2×2 matrix with the first index up and the second down). Here \mathcal{V}_j^a and $\mathcal{S}^{ab} = \mathcal{S}^{ba}$ are arbitrary and covariance allows us to write this as $\mathcal{V}_j^a = h_1 \hat{c}_j^a / \hat{r} - h_2 \hat{B}_j^a / \hat{r}^2 + h_3 \hat{c}_j^b \hat{c}_k^a / \hat{r}^3$ and $\mathcal{S}^{ab} = h_4 \delta^{ab} + h_5 \hat{c}_j^a \hat{c}_j^b / \hat{r}^2 + h_6 \hat{c}_j^a \hat{c}_j^d \hat{c}_k^d \hat{c}_k^b / \hat{r}^4$, where h_i depend on $\hat{r}^2 = (\hat{c}_j^a)^2$, $u = \hat{r}^{-4} (\hat{B}_j^a)^2$ and $v = \hat{r}^{-3} \det \hat{c}$. This gives a six dimensional matrix equation for the Hamiltonian which can be split into a “radial” and “angular” part ($\hat{H}_f = H_f / \hat{r}$),

$$H_{n'n}^{p'p} = \langle p', n' | H | p, n \rangle = E_p^{\ell_n} \delta_{nn'} \delta^{pp'} + \langle p', \ell_{n'} | \hat{r}^4 | p, \ell_n \rangle \langle n' | \frac{u}{2} | n \rangle + \langle p', \ell_{n'} | \hat{r} | p, \ell_n \rangle \langle n' | \hat{H}_f | n \rangle. \quad (3.1)$$

Our basis explicitly diagonalizes the angular part (“spherical harmonics”) in terms of invariant polynomials. In Fig. 2 the angular band structure is given by plotting $\langle n' | u | n \rangle \neq 0$ or $\langle n' | \hat{H}_f | n \rangle \neq 0$

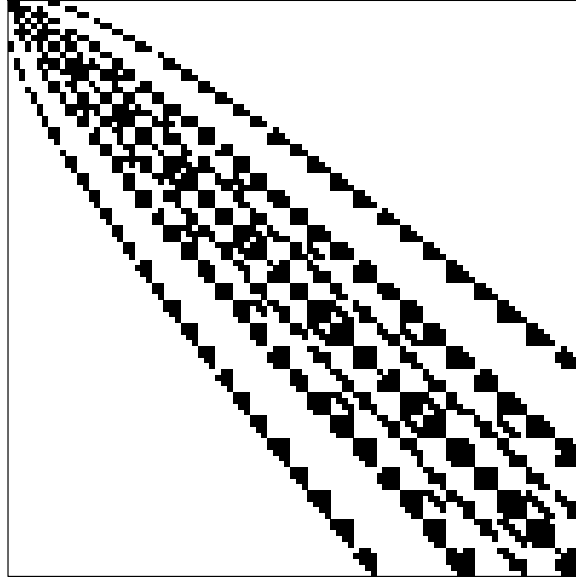


Figure 2: Band structure of the reduced Hamiltonian, for the first 100 states.

as black squares. The different bands can be traced to come from the selection rules $|\delta\ell| = 0, 2, 4$ for the matrix elements of u , and $|\delta\ell| = 1$ for the matrix elements of \hat{H}_f .

The radial wave function is $\phi_p^\ell(\hat{r}) = C_p^\ell \hat{r}^{-3} j_\ell(k_p^\ell \hat{r})$, with $E_p^\ell = \frac{1}{2} (k_p^\ell)^2$ and $\ell = 2L + 3$ ($j_\ell(z)$ is the spherical Bessel function and C_p^ℓ is the normalization constant). It diagonalizes the kinetic term, $-\frac{1}{2} \hat{r}^{-8} \partial_{\hat{r}} \hat{r}^8 \partial_{\hat{r}} + \hat{r}^{-2} L(2L + 7) = -\frac{1}{2} \hat{r}^{-3} [\hat{r}^{-2} \partial_{\hat{r}} \hat{r}^2 \partial_{\hat{r}} + \frac{1}{2} \hat{r}^{-2} \ell(\ell + 1)] \hat{r}^3$. To complete the construction of the basis, we need to address the question of boundary conditions (fixing the momenta k_p^ℓ), so that the groundstate wave function becomes a constant, after projecting to the vacuum valley. The vacuum valley is characterized by those configurations for which $u = 0$ (this implies $v = 0$), and \hat{r} measures the (three-dimensional) distance to the origin along this vacuum valley.

The wave function can be decomposed as $\Psi = \sum_{n=0}^{\infty} \hat{r}^{-3} f_n(\hat{r}) \chi_n(u, v; \hat{r})$, with $\chi_n(u, v; \hat{r})$ normalized eigenfunctions, so that $\langle \Psi | \Psi' \rangle = \sum_{n=0}^{\infty} \int 4\pi \hat{r}^2 d\hat{r} f_n^*(\hat{r}) f'_n(\hat{r})$. This means that we have to impose $\partial_{\hat{r}} f_0(\hat{r}) = 0$ at the boundary of the fundamental domain, $\hat{r} = b \equiv \pi g^{-2/3}(L)$. This is equivalent to $\langle \chi_0 | \partial_{\hat{r}}(\hat{r}^3 \Psi) \rangle = 0$, where the inner product is at fixed \hat{r} . That this boundary condition receives corrections is however an *artifact* of the truncation to the zero-momentum modes. If we take into account that in the full theory χ_0 also involves the non-zero momentum modes, the (gauge) symmetry guarantees[10] that *at the boundary* of the fundamental domain $\partial_{\hat{r}} f_0(b) = 0$, and this source of the breaking of supersymmetry is absent. Higher order terms in computing the effective Hamiltonian are required to deal with this[3]. We therefore impose $\partial_{\hat{r}}(\hat{r}^3 \Psi(\hat{r})) = 0$ at $\hat{r} = b \equiv \pi g^{-2/3}(L)$, knowing that it is only valid at small $g(L)$.

A direct measure for the failure of the Born-Oppenheimer approximation is given by $f^2(\hat{r}) - f_0^2(\hat{r}) \equiv \sum_{n=1}^{\infty} f_n^2(\hat{r})$, which as expected, deviates from zero when the gap is small, but *also* when we approach the boundary at $\hat{r} = b$ (due to the zero-momentum approximation). In Fig. 3 we show the results for $b = 4.4, 4.7$ and 5.0 , to illustrate that this second deviation decreases with increasing b (or decreasing coupling). This shows that the mismatch between the boundary condition and the truncation of the effective Hamiltonian does not affect the wave function in the neighborhood of the orbifold singularities, where the failure of the adiabatic approximation is non-perturbative.

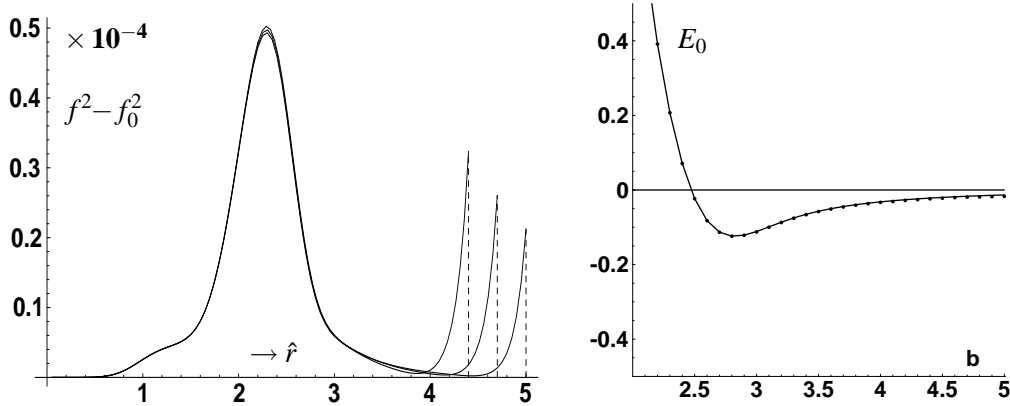


Figure 3: On the right is shown $f^2(\hat{r}) - f_0^2(\hat{r})$ extracted from the groundstate wave function Ψ_0 , satisfying the boundary condition $\partial_{\hat{r}}(\hat{r}^3 \Psi_0)(b) = 0$, for $b = 4.4, 4.7$ and 5.0 . We normalized with respect to $b = 5$, such that at $\hat{r} = 2$ all f^2 agree. On the left is shown the groundstate energy itself, using up to 20 radial modes for each of the 420 harmonics together with the lower bound from Temple's inequality (indicated by the dots).

Finally Fig. 3 also shows the groundstate energy E_0 (the curve gives the upper and the dots the lower bound). To keep the size of the matrix manageable, the components of the eigenvectors are removed when they are in absolute value below a threshold, without significantly affecting the accuracy. This process of *pruning* is performed iteratively, increasing the number of radial modes per angular state. Together with Temple's inequality[10] this is extremely efficient to optimize the accuracy and achieve numerical control.

We have only considered $SU(2)$ to illustrate how to go beyond the adiabatic approximation. Our analysis shows that, although the orbifold singularities are cause for concern, in the end they do not upset the result for the Witten index. For other groups this is much harder. Of course, a numerical analysis can never be entirely conclusive in deciding a theoretical issue

that involves the counting of *exact* zero-energy states. The computer code is still available at www.lorentz.leidenuniv.nl/vanbaal/susyYM (partly they could be used for other studies [11]).

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